A line source on an interface between two media

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(Received 20 August 1959)

The assumption of dynamic similarity is used to determine the velocity potential of an unsteady line source which lies on a plane separating two media of different density and sound velocity. The solution first derived is for a source whose strength varies with time as a step function; the pressure in this case may be identified with Hadamard's elementary solution which in a homogeneous fluid is $(c/2\pi)(c^2t^2-r^2)^{-\frac{1}{2}}$ if $r < ct$, and 0 if $r > ct$. We next derive the solution for a source whose strength has a delta-function time dependence; we then describe the results for a supersonic point source moving on the interface, and finally we transfer the results to solve the corresponding electromagnetic problem.

1. Introduction

In some recent work Craggs (1956, 1957) and Papadopoulos (1959*a*, 1959*b*) have shown the value of the assumption of dynamic similarity in the solution of a number of unsteady two-dimensional problems in various physical situations. In each case the unknown quantity satisfies the wave equation.

In this paper we shall determine the nature of the field of a uniform line source which is suddenly set up at some definite moment on the plane interface between two different homogeneous fluids. We assume a linearized equation of state, and that the source is weak enough for the acoustic approximation to be valid. We take the source to be at the origin $r = 0$, and we take the time $t = 0$ to be the moment at which the source is made active. Under the assumption that the subsequent unsteady motion is irrotational, it is well known (e.g. see Friedlander 1958) that the velocity potential satisfies the wave equation

$$\nabla^2\phi(r,\theta,t) = c^{-2}(\partial^2\phi/\partial t^2),$$

where c is the velocity of sound in the medium at rest, while the pressure change p and the particle velocity $\mathbf{q}$ satisfy the equations

$$p = \partial\phi/\partial t, \quad \rho\mathbf{q} = -\nabla\phi. \tag{1}$$

Here ρ refers to the constant density in an undisturbed medium.

Within a single uniform medium, it is known (e.g. see Lamb 1932) that the potential of a line source of uniform density $U(t)$ ($U(t) = 0$ if $t < 0$, $U(t) = 1$ if $t > 0$) is $(1/2\pi)\,\mathrm{sech}^{-1}(r/ct)$; it is clear from this result that it is reasonable to assume in the present problem that the velocity potential depends only on two variables s ($=r/t$) and θ. This is the assumption of dynamic similarity. It may be added that the pressure corresponding to the above potential is identical with Hadamard's elementary solution (1923) of the wave equation in two dimensions.

Suppose now that some quantity $S(s,\theta)$ satisfies the wave equation in the variables (r,θ,t). Then since $s = r/t$ it follows that S must satisfy the equation

$$s^2\left(1-\frac{s^2}{c^2}\right)\frac{\partial^2 S}{\partial s^2}+s\left(1-\frac{2s^2}{c^2}\right)\frac{\partial S}{\partial s}+\frac{\partial^2 S}{\partial \theta^2} = 0. \tag{2}$$

If $s > c$, the equation is hyperbolic. Put $s = c\sec u$, so that

$$\frac{\partial^2 S}{\partial \theta^2}-\frac{\partial^2 S}{\partial u^2} = 0, \tag{3}$$

and

$$s = f(u-\theta)+g(u+\theta), \tag{4}$$

where f and g are arbitrary functions, and the lines on which $u+\theta$ and $u-\theta$ are constant are characteristic lines tangent to the circle $s = c$. If $s < c$, equation (2) is elliptic. Put $s = c\,\mathrm{sech}\,(-v)$, so that

$$\frac{\partial^2 S}{\partial \theta^2}+\frac{\partial^2 S}{\partial v^2} = 0. \tag{5}$$

It follows from equation (5) that with the harmonic function S in the elliptic region we may introduce a conjugate $T(v,\theta)$, so that $W = S+iT$ is an analytic function, and such that

$$\frac{\partial S}{\partial v} = \frac{\partial T}{\partial \theta}, \quad \frac{\partial S}{\partial \theta} = -\frac{\partial T}{\partial v}. \tag{6}$$

In § 2 the detailed solution of the acoustic problem is given. In § 5 we describe the change to be made to give the results relevant in the setting up of a charged line or of a line current.

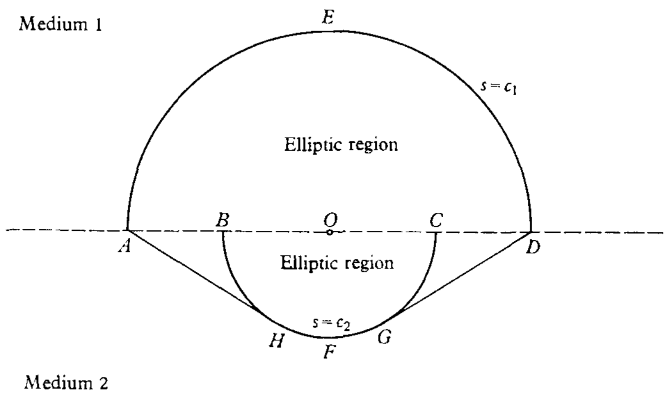

Figure 1. The elliptic regions in the (s, θ)-plane.

2. A line source of step function time dependence

In figure 1 we depict the physical situation in the (s,θ)-plane. The upper half of this plane, $0 < \theta < \pi$, represents the region occupied by a medium 1, and the lower half, $0 > \theta > -\pi$, that occupied by a medium 2. The respective density ρ and sound velocity c are distinguished by the suffix 1 or 2 where appropriate. We assume that $c_1 > c_2$. The semicircles $s = c_1$, $s = c_2$ separate the elliptic and the hyperbolic regions in each medium.

There are two requirements on the solution of our problem. The first is that the solution of the steady problem shall be approached in the limit as $s \to 0$ (i.e. as $t \to \infty$). The second is that at the interface both the pressure and the normal component of velocity shall be continuous.

Suppose that the quantity S and the velocity potential are related by the equation $\rho c^2 S = \phi$. From equations (1) it follows that the radial and transverse components (q_r, q_θ) of the velocity, and the pressure change p, satisfy the equations

$$\left.\begin{aligned} \frac{\partial S}{\partial s} &= -\frac{tq_r}{c^2} = -\frac{pt}{\rho c^2 s}, \\ \frac{\partial S}{\partial \theta} &= -\frac{rq_\theta}{c^2}. \end{aligned}\right\} \tag{7}$$

Putting $m = c_2/c_1$, $k = \rho_2/\rho_1$, then we may write the continuity conditions in the form

$$m^2 k \frac{\partial S_2}{\partial s} = \frac{\partial S_1}{\partial s}, \tag{8}$$

and

$$m^2 \frac{\partial S_2}{\partial \theta} = \frac{\partial^2 S_1}{\partial \theta}. \tag{9}$$

We may refer again to figure 1 to discuss some of the properties of the solution. In the hyperbolic region, it is clear that the value of S as $s \to \infty$ corresponds to the initial value of S. Hence S is uniformly zero at infinity for all values of θ, and from the nature of the solution (4) it follows that the value of S is everywhere zero outside the region $AFDE$. Within the triangle CDG the solution is necessarily of the form $S_2 = f(u-\theta)$, and in the triangle ABH the solution must be of the form $S_2 = g(u+\theta)$, where f and g are functions to be determined. The solution must be symmetric about the vertical axis in figure 1; hence we need only examine the field in the right-hand half of the (s, θ)-plane.

Consider the region $s \leqslant c_1$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. In this region we have that (i) the line OE is a line of symmetry on which $\partial S_1/\partial\theta = 0$. The arc ED, which is the envelope of the characteristics in the hyperbolic region, is itself a characteristic. Across this arc the pressure and the radial velocity will be discontinuous. The tangential velocity component must be continuous, however, so that (ii) $\partial S_1/\partial\theta = 0$ on ED.

The continuity of pressure and of normal velocity across the interface CD implies that on CD

$$\frac{\partial S_1}{\partial s} = m^2 k \frac{\partial S_2}{\partial s} = m^2 k \frac{\partial S_2}{\partial u_2}\frac{\partial u_2}{\partial s} = -m^2 k \frac{\partial u_2}{\partial s}\frac{\partial S_2}{\partial \theta} = -k\frac{\partial u_2}{\partial s}\frac{\partial S_1}{\partial \theta} = k\frac{\partial u_2}{\partial s}\frac{\partial s}{\partial v_1}\frac{\partial T_1}{\partial s}.$$

Thus

$$\frac{\partial}{\partial s}\left\{S_1 - kT_1\frac{\partial u_2}{\partial v_1}\right\} = 0,$$

so that

$$\mathscr{R}\left\{\frac{\partial W_1}{\partial s}\left(1 + ik\frac{\partial u_2}{\partial v_1}\right)\right\} = 0. \tag{10}$$

On physical grounds we may expect singularities in W_1 only at the points O, C, D, E. The first quadrant in the circle in the (s, θ)-plane corresponds to a semi-infinite strip in the complex (v_1, θ)-plane and under the transformation

$$\zeta_1 = \xi_1 + i\eta_1 = \operatorname{sech}(v_1 + i\theta)$$

we may map this strip conformally into the upper half of the complex ζ_1-plane. The conditions just enumerated are that

(i) singularities are to be expected only at the points $\zeta_1 = 0$ m, 1 and ∞;
(ii) $\partial W_1/\partial\zeta_1$ is imaginary, for $\xi_1 = 0$ $\eta_1 > 0$;
(iii) $\partial W_1/\partial\zeta_1$ is imaginary, for $\eta_1 = 0$ $|\xi_1| > 1$; and
(iv) for $\eta_1 = 0$, $m < \xi_1 < 1$,

$$\frac{\partial W_1}{\partial\zeta_1} = R(\zeta_1)\left\{1 - \frac{i}{mk}\left(\frac{\zeta_1^2 - m^2}{1-\zeta_1^2}\right)^{\frac{1}{2}}\right\}^{-1},$$

where $R(\zeta_1)$ is a function which must take real values on this segment of the real axis. Since $s = c_1$, $\theta = \frac{1}{2}\pi$ is an ordinary point both for S and for $\partial S/\partial\theta$, it follows that as $|\zeta_1| \to \infty$,

(v) $\partial W_1/\partial\zeta_1 = O(\zeta_1^{-2-\delta})$ with $\delta > 0$. As $\zeta_1 \to 0$, the field must approach the steady state value, and therefore

(vi) $\partial W_1/\partial\zeta_1 = O(\zeta_1^{-1})$ for $\zeta_1 \to 0$.

Thus, after applying these conditions, we may write

$$\frac{\partial W_1}{\partial\zeta_1} = \frac{F(\zeta_1)}{\zeta_1[(1-\zeta_1^2)^{\frac{1}{2}} - (i/mk)(\zeta_1^2 - m^2)^{\frac{1}{2}}]}, \tag{11}$$

it being implied that $F(\zeta_1)$ is bounded as $|\zeta_1| \to \infty$, is real on the positive real axis for $\xi_1 > m$, and is real on the imaginary axis. This final condition implies that F must be an even function of ζ_1.

Whatever may be the formula for $F(\zeta_1)$ which we shall determine, we must, in setting up the solution for medium 2, satisfy the continuity conditions across OC. If in this elliptic region we use the conformal mapping $\zeta_2 = \xi_2 + i\eta_2 = \operatorname{sech}(v_2 + i\theta)$ to bring the region of interest, $s < c_2$, $0 > \theta > -\frac{1}{2}\pi$, into the fourth quadrant of the ζ_2-plane, then across OC, $\zeta_1 = m\zeta_2$, and the continuity conditions are

$$\left.\begin{aligned} m^2k\frac{\partial S_2}{\partial\zeta_2} &= \frac{\partial S_1}{\partial\zeta_2}, \\ \text{and} \qquad m^2k\frac{\partial T_2}{\partial\zeta_2} &= k\left(\frac{1-m^2\zeta_2^2}{1-\zeta_2^2}\right)^{\frac{1}{2}}\frac{\partial T_1}{\partial\zeta_2}. \end{aligned}\right\} \tag{12}$$

It is clear under the conditions imposed that $F(\zeta_1)$ must be real on the whole of the real axis. If $F(\zeta_1)$ is complex on the real axis for $0 < \xi_1 < m$, there must be branch points of F at the points $\xi_1 = 0$ and $\xi_1 = m$. Hence for this region we may write

$$F(\zeta_1) = A(\zeta_1) + iB(\zeta_1)\left(\frac{\zeta^2}{m^2 - \zeta^2}\right)^{\frac{1}{2}}, \tag{13}$$

where $A(\zeta_1)$ and $B(\zeta_1)$ are even functions of ζ_1 which are real on the whole of the real axis and which are bounded as $|\zeta_1| \to \infty$. The continuity conditions (12) lead to the equation

$$m^2k\frac{\partial W_2}{\partial\zeta_2} = \frac{A(m\zeta_2) + ikB(m\zeta_2)[(1-m^2\zeta_2^2)\zeta_2^2/(1-\zeta_2^2)]^{\frac{1}{2}}}{\zeta_2\{(1-m^2\zeta_2^2)^{\frac{1}{2}} + k^{-1}(1-\zeta_2^2)^{\frac{1}{2}}\}}. \tag{14}$$

This result, derived for real values of ζ_2 with $0 < \zeta_2 < 1$, may be continued analytically into the whole of the fourth quadrant of the ζ_2-plane. If the function B exists, expression (14) has a simple pole at $\zeta_2 = 1$, so that W_2 has a discontinuity at this point. At the corresponding point $\zeta_1 = m$, W_1 has no discontinuity (from equations (11) and (13)); the only way to avoid this inconsistency is to set $B \equiv 0$. It follows that $F(\zeta_1)$ is a regular function bounded at infinity and real on the real axis, so that it can only be a real constant A. The vanishing of B means that there is no normal velocity between the two subsonic regions. This section OC of the interface is a vortex sheet which is steadily lengthening.

We may now write the explicit results

$$\left.\begin{aligned} \frac{\partial W_1}{\partial \zeta_1} &= \frac{A}{\zeta_1\left\{(1-\zeta_1^2)^{\frac{1}{2}} - \dfrac{i}{mk}(\zeta_1^2 - m^2)^{\frac{1}{2}}\right\}}, \\ \text{and} \qquad m^2 k \frac{\partial W_2}{\partial \zeta_2} &= \frac{A}{\zeta_2\{(1-m^2\zeta_2^2)^{\frac{1}{2}} + k^{-1}(1-\zeta_2^2)^{\frac{1}{2}}\}}. \end{aligned}\right\} \tag{15}$$

The velocity components and the pressure, which are related to the derivatives of S as in equations (7), may now be found. Thus, for $s < c_1$ in medium 1, when $\zeta_1 = \operatorname{sech}(v_1 + i\theta)$ and $s = c_1 \operatorname{sech}(-v_1)$,

$$\left.\begin{aligned} \frac{\partial S_1}{\partial s} &= -\frac{A}{s\left(1-\dfrac{s^2}{c_1^2}\right)^{\frac{1}{2}}} \mathscr{R}\left\{\frac{1}{1 - \dfrac{i}{mk}\left(\dfrac{\zeta_1^2 - m^2}{1-\zeta_1^2}\right)^{\frac{1}{2}}}\right\}, \\ \frac{\partial S_1}{\partial \theta} &= A\mathscr{I}\left\{\frac{1}{1 - \dfrac{i}{mk}\left(\dfrac{\zeta_1^2 - m^2}{1-\zeta_1^2}\right)^{\frac{1}{2}}}\right\}, \end{aligned}\right\} \tag{16}$$

and for $s < c_2$ in medium 2, when $\zeta_2 = \operatorname{sech}(v_2 + i\theta)$ and $s = c_2 \operatorname{sech}(-v_2)$

$$\left.\begin{aligned} m^2 k \frac{\partial S_2}{\partial s} &= -\frac{A}{s\left(1-\dfrac{s^2}{c_2^2}\right)^{\frac{1}{2}}} \mathscr{R}\left\{\frac{1}{\dfrac{1}{k} + \left(\dfrac{1-m^2\zeta_2^2}{1-\zeta_2^2}\right)^{\frac{1}{2}}}\right\}, \\ m^2 k \frac{\partial S_2}{\partial \theta} &= A\mathscr{I}\left\{\frac{1}{\dfrac{1}{k} + \left(\dfrac{1-m^2\zeta_2^2}{1-\zeta_2^2}\right)^{\frac{1}{2}}}\right\}. \end{aligned}\right\} \tag{17}$$

The constant A is a measure of the volume of fluid produced by the line source; by considering the steady state in the limit $s \to 0$ or $\zeta \to 0$, we find that the volume created in each medium is $\pi k A c_1^2/(1+k)$ in medium 1 and $\pi A c_1^2/(1+k)$ in medium 2. Thus the strength, for $t > 0$, of the source is Ac_1^2; the fluid then produced is apportioned between the two media in the inverse ratio of the densities, so that the mass of fluid produced is the same in each medium.

To determine the velocity components and the pressure in the hyperbolic region CDG, we use the explicit results derived from equation (16) and (17) for points on the boundary CD, and we use the characteristic form of the solution in CDG to find the complete result. Thus in CDG $S_2 = f(u-\theta)$ where $s/c_2 = \sec u$; it follows that $\partial S_2/\partial\theta = -f'(u-\theta)$. For $c_2 < s < c_1$,

$$\begin{aligned} f'(u) = -\frac{1}{m^2}\left(\frac{\partial S_1}{\partial \theta}\right)_{\theta=0} &= -\frac{A}{m^3 k}\left(\frac{s^2-c_2^2}{c_1^2-s^2}\right)^{\frac{1}{2}} \Bigg/ \left\{1 + \frac{1}{m^2k^2}\left(\frac{s^2-c_2^2}{c_1^2-s^2}\right)^{\frac{1}{2}}\right\} \\ &= H(s), \text{ say.} \end{aligned} \tag{18}$$

It follows that in CDG,

$$\left.\begin{aligned} \partial S_2/\partial\theta &= -H(s^*),\\ \text{where}\qquad s^*/c_2 &= \sec(u-\theta) = s/c_2\cos\theta+\sin\theta(s^2-c_2^2)^{\frac{1}{2}}.\end{aligned}\right\} \tag{19}$$

Similarly the derivative $\partial S_2/\partial s$ is given by the equation

$$\frac{\partial S_2}{\partial s} = \frac{H(s^*)}{s[(s^2/c_2^2)-1]^{\frac{1}{2}}}. \tag{20}$$

3. A line source with delta-function strength

As far as fluid motion is concerned the analysis of §2 is merely an exercise in setting up a quantity which has the property of dynamic similarity in a region with the properties given. By assuming uniform densities for two media, we are of course neglecting gravity, but it is not clear whether we can neglect the effect at the interface. Under the usual first-order approximations, given a small displacement $y = \eta$ in the position of the surface, the conditions of continuity of pressure and of normal velocity take the form

$$\rho_1\frac{\partial\phi_1}{\partial t}-\rho_2\frac{\partial\phi_2}{\partial t} = g(\rho_1-\rho_2)\,\eta+\ldots,$$

$$\frac{1}{\rho_1}\frac{\partial\phi_1}{\partial y}-\frac{1}{\rho_2}\frac{\partial\phi_2}{\partial y} = \frac{\partial\eta}{\partial t},$$

where y is the axis normal to the interface.

Although from these equations alone we may find only what sort of surface waves may exist on the interface by prescribing a form for the displacement, we shall eliminate the dispersive effects due to gravity by insisting that η be a function continuous in x and t, small in comparison with the quantities

$$[p_0+p_1]\,[g(\rho_1-\rho_2)]^{-1} \quad\text{and}\quad [p_0+p_2]\,[g(\rho_1-\rho_2)]^{-1},$$

where p_0 is the steady pressure at the interface. Then $\partial\eta/\partial t$ is also small. The assumption of an acoustic line source of infinitesimal amplitude and step function time-dependence in §2 in no way violates this assertion.

With these remarks in mind we can now state that the results for an (acoustic) source of delta-function time-dependence on the interface are obtained from the formulae in §2 simply by differentiating with respect to time throughout. Thus the known results for $\partial\phi/\partial t$ in §2 represent the values of the velocity potential in the impulse problem. These values are

$$\phi_1 = \frac{A\rho_1 c_1^2}{[t^2-(r^2/c_1^2)]^{\frac{1}{2}}}\mathscr{R}\left[1-\frac{i}{mk}\left(\frac{\zeta_1^2-m^2}{1-\zeta_1^2}\right)^{\frac{1}{2}}\right]^{-1} \quad\text{for}\quad r < c_1 t,$$

where
$$\zeta_1 = \frac{r}{c_1 t\cos\theta - i\sin\theta(c_1^2t^2-r^2)^{\frac{1}{2}}},$$

and
$$\phi_2 = \frac{A\rho_1 c_1^2}{[t^2-(r^2/c_2^2)]^{\frac{1}{2}}}\mathscr{R}\left[\frac{1}{k}+\left(\frac{1-m^2\zeta_2^2}{1-\zeta_2^2}\right)^{\frac{1}{2}}\right]^{-1}, \quad\text{for}\quad r < c_2 t,$$

where
$$\zeta_2 = \frac{r}{c_2 t\cos\theta - i\sin\theta(c_2^2t^2-r^2)^{\frac{1}{2}}}.$$

Also
$$\phi_2 = \frac{A\rho_1 c_1^2}{[(r^2/c^2)-t^2]^{\frac{1}{2}}}\left[\frac{1}{m}\left\{\frac{c_0^2-c_2^2}{c_1^2-c_0^2}\right\}^{\frac{1}{2}}\Big/\left\{1+\frac{1}{m^2k^2}\left(\frac{c_0^2-c_2^2}{c_1^2-c_0^2}\right)\right\}\right],$$

where
$$c_0 = \frac{rc_2}{c_2 t\cos\theta+\sin\theta(r^2-c_2^2t^2)^{\frac{1}{2}}}.$$

This final expression for ϕ is valid within the hyperbolic region GCD.

4. The supersonic source

The results given in §2 are immediately applicable in the calculation of the fields which accompany a semi-infinite line source of uniform strength, which is moving lengthways on the interface between two fluids. This line singularity moves steadily with a velocity V which must be supersonic with respect to both media, and it lies on the z-axis.

The flow is conical, and the variable s is now of the form $s = rV/(Vt-z)$ for steady motion in the positive z-direction. The s- and θ-derivatives of the potential are then given by equations (17) to (20) if we replace c_1 by $c_1V(V^2-c_1^2)^{-\frac{1}{2}}$ and c_2 by $c_2V(V^2-c_2^2)^{-\frac{1}{2}}$, and if we modify the quantity m accordingly. The z-derivative, that is the quantity $\{s/(Vt-z)\}(\partial S/\partial s)$, then represents either the velocity component parallel to the line singularity, or the potential of a supersonic point source moving along the interface.

5. The electromagnetic problem

The results derived in the acoustic problem are applicable in the theory of electromagnetic pulses involved in the sudden setting-up of a current in an infinite line or of a charged line on the interface between two media. In the former case the vector potential has only one component $A_z = c^2S(s,\theta)$, the constant k is the ratio of the magnetic permeabilities μ_2/μ_1, and m is the ratio c_2/c_1. The non-zero field components, derived from Maxwell's equations, are

$$B_r = \frac{\mu c^2}{r}\frac{\partial S}{\partial\theta}, \quad B_\theta = -\frac{\mu c^2}{t}\frac{\partial S}{\partial s} \quad \text{and} \quad E_z = -sB_\theta.$$

For the charged line we relate the quantity S to the scalar potential Φ through the equation $\Phi = c^2S$. In this case k is the ratio of the dielectric constants ϵ_2/ϵ_1, and the non-zero field components are

$$E_r = -\frac{c^2}{t}\frac{\partial S}{\partial s}, \quad E_\theta = \frac{c^2}{t}\frac{\partial S}{\partial\theta} \quad \text{and} \quad B_z = \frac{1}{t}\frac{\partial S}{\partial\theta}.$$

The results given in equations (16), (17), (19) and (20) may be used directly to derive the field components.

6. Conclusion

The assumption of dynamic similarity is used to determine the velocity potential and pressure field of an impulsive line source which is suddenly set up on the plane which separates two media of different density and sound velocity.

The solution for the potential may be identified with Hadamard's elementary solution of the wave equation in the homogeneous case, when

$$\phi = \frac{c}{2\pi}(c^2t^2 - r^2)^{-\frac{1}{2}} \quad \text{if} \quad r < ct,$$
$$\phi = 0 \quad \text{if} \quad r > ct.$$

In the case of two media considered in this paper, there are similar algebraic singularities on the shock fronts $r = ct$ in medium 1 and $r = c_2 t$ in medium 2.

A feature of the solution is that since $\partial W/\partial \zeta$ is real on the section OC of the interface, there is no normal velocity between the two subsonic regions. This section of the interface is a contact discontinuity (i.e. OC is a steadily expanding vortex-sheet).

The author presented these results verbally at the spring (1959) meeting of the U.R.S.I. in Washington, D.C. A similar method has been used independently by Keller & Gardner (1959) to find the solution for a line dipole. This research has been sponsored in part by Air Force Cambridge Research Center under contract AF 19(604)–4561, and in part by the Office of Naval Research and the David W. Taylor Model Basin under contract Nonr-562(24).

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